

# A MURPHY BASIS OF $q$ -BRAUER ALGEBRAS

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**Abstract.** A new basis of the  $q$ -Brauer algebra is introduced, which is a lift of the Murphy basis of the Hecke algebra of the symmetric group. This basis is a cellular basis in the sense of Graham and Lehrer. Subsequently, using combinatorial language we prove that the non-isomorphic simple  $q$ -Brauer modules are indexed by the  $e(q^2)$ -restricted partitions of  $n - 2k$  where  $k$  is an integer,  $0 \leq k \leq [n/2]$ . When the  $q$ -Brauer algebra has low-dimension a criterion of semisimplicity is given, which is used to show that the  $q$ -Brauer algebra is in general not isomorphic to the BMW-algebra.

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## 1. INTRODUCTION

In the classical Schur-Weyl duality the actions of the general linear group  $GL(N)$  and the symmetric group  $S_n$  on the tensor power spaces  $(\mathbb{C}^N)^{\otimes n}$  are centralizers of each other. In 1937, Richard Brauer showed that when replacing  $GL(N)$  by the orthogonal subgroup  $O(N)$  or the symplectic subgroup  $Sp(N)$  the corresponding centralizer is a larger algebra containing the symmetric group, called the Brauer algebra  $D_n(N)$ . In the quantum case, there is an analogue of these dualities in which:  $GL(N)$  and  $S_n$  are substituted by the quantized enveloping algebra  $U_q(\mathfrak{gl}_N)$  and the Hecke algebra of the symmetric group  $H_n(q)$  respectively (see Jimbo [14] and Le Duc and Ram [16]);  $O(N)$  (resp.  $Sp(N)$ ) and  $D_n(N)$  are substituted by the quantized enveloping algebras  $U_q(\mathfrak{o}_N)$  (resp.  $U_q(\mathfrak{sp}_N)$ ) and the BMW-algebra  $\mathfrak{B}_n$ , a  $q$ -deformation of the Brauer algebra, with appropriate choices of parameters respectively ([3], Section 10.2).

Recently, another  $q$ -deformation of the Brauer algebra has been introduced by Wenzl [22] via generators and relations who called it the  $q$ -Brauer algebra. This algebra contains

the Hecke algebra of the symmetric group as a subalgebra and, over the field  $\mathbb{Q}(r, q)$ , is semisimple and isomorphic to the Brauer algebra. Some applications of this algebra were found by Wenzl in [23] and [24]. In [7] Dung showed that the generic  $q$ -Brauer algebra is cellular, and in particular it is an iterated inflation of Hecke algebras of the symmetric group.

The subjects of this note are following two questions.

**Question 1.** How to give a combinatorial and direct proof for parametrization of simple modules of the  $q$ -Brauer algebra shown in [7]?

**Question 2.** In general, does there exist an algebra isomorphism between the  $q$ -Brauer algebra and the BMW-algebra?

In [7], Dung showed that the simple  $q$ -Brauer modules up to isomorphism are indexed by the  $e(q^2)$ -restricted partitions of  $n - 2k$  where  $k$  is an integer,  $0 \leq k \leq [n/2]$ . However, the proof needs to use the structure "iterated inflation" of the  $q$ -Brauer algebra which is complicated. In this article, we give a simple answer for the question 1 in Theorem 4.1 via using the combinatorial language which does not relate to the structure of the  $q$ -Brauer algebra. Then question 2 is fully answered by giving a criterion for semisimplicity of the  $q$ -Brauer algebra,  $Br_n(r^2, q^2)$ , in the case  $n \in \{2, 3\}$  and some explicit calculations in Propositions 5.1, 5.2 and Examples 5.5, 5.6, 5.7. The statement is that:

**Claim 1.1.** *In general, there does not exist an algebra isomorphism between the  $q$ -Brauer algebra  $Br_n(r^2, q^2)$  (resp.  $Br_n(r, q)$ ) and the BMW-algebra  $\mathcal{B}_n$ .*

To obtain the results above, we first introduce a new basis of the  $q$ -Brauer algebra that is a lift of the Murphy bases of the Hecke algebra of the symmetric groups ([17] or [18]) and the BMW-algebra [8]. The main result stated in Theorem 3.10 is that the  $q$ -Brauer algebra over a commutative ring has the basis consisting of elements that are indexed by two pairs, in each pair the first entry is a standard tableaux and the second one is a certain partial Brauer diagram. This basis is a cellular basis and exists for every version (one or two parameters) of the  $q$ -Brauer algebra over a field of any characteristic. Then, we can apply the theory of cellular algebra to produce cell (Specht) and simple modules of the  $q$ -Brauer algebra over a field of any characteristic. This provides a combinatorial approach to study the representation theory of the  $q$ -Brauer algebra like that of the Hecke algebra of the symmetric group and its  $q$ -Schur algebra or the cyclotomic  $q$ -Schur algebra or the BMW-algebra.

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## 2. NOTATION AND PRELIMINARIES

This section recalls the concepts tableaux and Young subgroup, and it collects basic and necessary facts of the representation theory of the Hecke of the symmetric group. We introduce these with a slight difference in which the usual symmetric group and its deformation, the Hecke algebra, are replaced by isomorphic ones, written in a different

way. In particular, we need to use background on a subgroup of the symmetric group and the representation theory of its corresponding Hecke algebra. However, the usual results in the literature hold true for this restriction (see [4], [17] or [18]).

**2.1. Combinatorics.** Throughout,  $n$  will denote a positive integer, and  $S_n$  will be the symmetric group acting on  $\{1, \dots, n\}$  on the right. For  $i$  an integer,  $1 \leq i < n$ , let  $s_i$  denote the transposition  $(i, i+1)$ . Then  $S_n$  is generated by  $s_1, s_2, \dots, s_{n-1}$ , which satisfy the defining relations

$$\begin{aligned} s_i^2 &= 1 & \text{for } 1 \leq i < n; \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} & \text{for } 1 \leq i < n-1; \\ s_i s_j &= s_j s_i & \text{for } 2 \leq |i-j|. \end{aligned}$$

Let  $k$  be an integer,  $0 \leq k \leq [n/2]$ . Denote  $S_{2k+1, n}$  to be the subgroup of  $S_n$  generated by generators  $s_{2k+1}, s_{2k+2}, \dots, s_{n-1}$ . This subgroup is isomorphic to a symmetric group  $S_{n-2k}$ .

An expression  $w = s_{i_1} s_{i_2} \dots s_{i_m}$  in which  $m$  is minimal is called a *reduced expression* for  $w$ , and  $\ell(w) = m$  is the *length* of  $w$ .

Let  $k$  be an integer,  $0 \leq k \leq [n/2]$ . If  $n - 2k > 0$ , a *partition* of  $n - 2k$  is a sequence  $\lambda = (\lambda_1, \lambda_2, \dots)$  of non-negative integers such that  $\lambda_i \geq \lambda_{i+1}$  for all  $i \geq 1$  and  $|\lambda| = \sum_{i=1} \lambda_i = n - 2k$ . The integers  $\lambda_i$ , for  $i \geq 1$ , are the parts of  $\lambda$ ; if  $\lambda_i = 0$  for  $i > m$  we identify  $\lambda$  with  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  and denote  $\lambda \vdash n - 2k$ . If  $n - 2k = 0$ , write  $\lambda = \emptyset$  for the empty partition. The *Young diagram* of a partition  $\lambda$  is the subset

$$[\lambda] = \{(i, j) : \lambda_i \geq j \geq 1 \text{ and } i \geq 1\} \subseteq \mathbb{N} \times \mathbb{N}.$$

The elements of  $[\lambda]$  are the *nodes* of  $\lambda$  and more generally a node is a pair  $(i, j) \in \mathbb{N} \times \mathbb{N}$ . The diagram  $[\lambda]$  is represented as an array of boxes with  $\lambda_i$  boxes on the  $i$ -th row. For example, if  $\lambda = (3, 1)$ , then  $[\lambda] = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$ . Let  $k$  be an integer,  $0 \leq k \leq [n/2]$ , and  $\lambda$  be a partition of  $n - 2k$ . A  $\lambda$ -tableau labeled by  $\{2k+1, 2k+2, \dots, n\}$  is a bijection  $\mathbf{t}$  from the nodes of the diagram  $[\lambda]$  to the integers  $\{2k+1, 2k+2, \dots, n\}$ . A given  $\lambda$ -tableau  $\mathbf{t} : [\lambda] \rightarrow \{2k+1, 2k+2, \dots, n\}$  can be visualized by labeling the nodes of the diagram  $[\lambda]$  with the integers  $2k+1, 2k+2, \dots, n$ . For instance, if  $n = 10$ ,  $k = 2$  and  $\lambda = (3, 2, 1)$ ,

$$(2.1) \quad \mathbf{t} = \begin{array}{|c|c|c|} \hline 5 & 7 & 8 \\ \hline 6 & 10 & \\ \hline 9 & & \\ \hline \end{array}$$

represents a  $\lambda$ -tableau. A  $\lambda$ -tableau  $\mathbf{t}$  labeled by  $\{2k+1, 2k+2, \dots, n\}$  is said to be *standard* if the entries in  $\mathbf{t}$  increase from left to right in each row and from top to bottom in each column. Let  $\mathbf{t}^\lambda$  denote the  $\lambda$ -tableau in which the integers  $2k+1, 2k+2, \dots, n$  are entered in increasing order from left to right along the rows of  $[\lambda]$ . For instance, let  $n = 10$ ,  $k = 2$  and  $\lambda = (3, 2, 1)$ ,

$$\mathbf{t}^\lambda = \begin{array}{|c|c|c|} \hline 5 & 6 & 7 \\ \hline 8 & 9 & \\ \hline 10 & & \\ \hline \end{array}.$$

The tableau  $\mathbf{t}^\lambda$  is referred to as the *superstandard tableau*. Denote  $\text{Std}(\lambda)$  the set of standard  $\lambda$ -tableaux labeled by the integers  $\{2k+1, 2k+2, \dots, n\}$ .

For  $\lambda$  and  $\mu$  arbitrary partitions, the *dominance order* on partitions is defined as follows:  $\lambda \supseteq \mu$  if either

- (1)  $|\mu| > |\lambda|$  or
- (2)  $|\mu| = |\lambda|$  and  $\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m \mu_i$  for all  $m > 0$ .

We will write  $\lambda \triangleright \mu$  to mean that  $\lambda \supseteq \mu$  and  $\lambda \neq \mu$ . The symmetric group  $S_{2k+1,n}$  acts on the set of  $\lambda$ -tableaux on the right in the usual manner, by permuting the integer labels of the nodes of  $[\lambda]$ . For example,

$$(2.2) \quad \begin{array}{|c|c|c|} \hline 5 & 6 & 7 \\ \hline 8 & 9 & \\ \hline 10 & & \\ \hline \end{array} (6, 8, 7)(9, 10) = \begin{array}{|c|c|c|} \hline 5 & 7 & 8 \\ \hline 6 & 10 & \\ \hline 9 & & \\ \hline \end{array}.$$

Let  $\lambda$  be a partition of  $n - 2k$ , define *Young subgroup*  $S_\lambda$  to be the row stabilizer of  $\mathbf{t}^\lambda$  in  $S_{2k+1,n}$ . For instance, when  $n = 10$ ,  $k = 2$  and  $\lambda = (3, 2, 1)$ , then a direct calculation yields  $S_\lambda = \langle s_5, s_6, s_8 \rangle$ . To each  $\lambda$ -tableau  $\mathbf{t}$ , associate a unique permutation  $d(\mathbf{t}) \in S_{2k+1,n}$  by the condition  $\mathbf{t} = \mathbf{t}^\lambda d(\mathbf{t})$ . Using the tableau  $\mathbf{t}$  in (2.1) above it deduces that  $d(\mathbf{t}) = (6, 8, 7)(9, 10)$  by (2.2).

**2.2. The Hecke algebra of the symmetric group.** Let  $R$  denote an integral domain and  $q$  be a unit in  $R$ . The Hecke algebra (over  $R$ ) of the symmetric group  $S_{2k+1,n}$  is the unital associative  $R$ -algebra  $H_{2k+1,n}(q^2)$  with generators  $g_{2k+1}, g_{2k+2}, \dots, g_{n-1}$ , which satisfy the defining relations

$$\begin{aligned} g_i^2 &= (q^2 - 1)g_i + q^2 && \text{for } 2k+1 \leq i < n; \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} && \text{for } 2k+1 \leq i < n-1; \\ g_i g_j &= g_j g_i && \text{for } 2 \leq |i - j|. \end{aligned}$$

Note that  $H_{2k+1,n}(q^2)$  is isomorphic to the usual Hecke algebra  $H_{n-2k}(q^2)$  in the literature. If  $w \in S_{2k+1,n}$  and  $s_{i_1} s_{i_2} \cdots s_{i_m}$  is a reduced expression for  $w$ , then  $g_w = g_{i_1} g_{i_2} \cdots g_{i_m}$  is a well defined element of  $H_{2k+1,n}(q^2)$  and the set  $\{g_w : w \in S_{2k+1,n}\}$  freely generates  $H_{2k+1,n}(q^2)$  as an  $R$ -module (Theorems 1.8 and 1.13 of [17]). From now on, we abbreviate  $H_{2k+1,n}$  replacing  $H_{2k+1,n}(q^2)$ .

In the following we collect standard facts from the representation theory of the Hecke algebra of the symmetric group; details can be found in [17] or [18]. If  $\mu$  is a partition of  $n - 2k$ , define the element

$$(2.3) \quad c_\mu = \sum_{\sigma \in S_\mu} g_\sigma.$$

Let  $*$  denote the algebra involution of  $H_{2k+1,n}$  mapping  $g_w \mapsto g_w^* := g_{w^{-1}}$  for  $w \in S_{2k+1,n}$ . Denote  $\check{\mathcal{H}}_{2k+1,n}^\lambda$  to be the  $R$ -module in  $H_{2k+1,n}$  with basis

$$(2.4) \quad \{c_{\mathbf{s}\mathbf{t}} = g_{d(\mathbf{s})}^* c_\mu g_{d(\mathbf{t})} : \mathbf{s}, \mathbf{t} \in \text{Std}(\mu), \text{ where } \mu \triangleright \lambda\}.$$

The next statement is due to Murphy in [18].

**Theorem 2.1.** *The Hecke algebra  $H_{2k+1,n}$  is free as an  $R$ -module with basis*

$$(2.5) \quad \mathcal{M} = \left\{ c_{\mathbf{s}\mathbf{t}} = g_{d(\mathbf{s})}^* c_\lambda g_{d(\mathbf{t})} \mid \begin{array}{l} \text{for } \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ and} \\ \lambda \text{ a partition of } n - 2k \end{array} \right\}.$$

Moreover, the following statements hold.

- (1) *The  $R$ -linear involution  $*$  satisfies  $*$  :  $c_{\mathbf{s}\mathbf{t}} \mapsto c_{\mathbf{t}\mathbf{s}}$  for all  $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$ .*

- (2) Suppose that  $h \in H_{2k+1,n}$ , and that  $\mathfrak{s}$  is a standard  $\lambda$ -tableau. Then there exist  $a_{\mathfrak{t}} \in R$ , for  $\mathfrak{t} \in \text{Std}(\lambda)$ , such that for all  $\mathfrak{s} \in \text{Std}(\lambda)$ ,

$$(2.6) \quad c_{\mathfrak{s}\mathfrak{v}}h \equiv \sum_{\mathfrak{t} \in \text{Std}(\lambda)} a_{\mathfrak{t}}c_{\mathfrak{st}} \pmod{\check{\mathcal{H}}_{2k+1,n}^{\lambda}}.$$

The basis  $\mathcal{M}$  is cellular in the sense of [9]. If  $\lambda$  is a partition of  $n - 2k$ , the *cell (or Specht) module*  $S^{\lambda}$  for  $H_{2k+1,n}$  is the  $R$ -module freely generated by

$$(2.7) \quad \{c_{\mathfrak{s}} = c_{\lambda}g_{d(\mathfrak{s})} + \check{\mathcal{H}}_{2k+1,n}^{\lambda} : \mathfrak{s} \in \text{Std}(\lambda)\},$$

and with the right  $H_{2k+1,n}$ -action

$$(2.8) \quad c_{\mathfrak{s}}h = \sum_{\mathfrak{t} \in \text{Std}(\lambda)} a_{\mathfrak{t}}c_{\mathfrak{t}}, \quad \text{for } h \in H_{2k+1,n},$$

where the coefficients  $a_{\mathfrak{t}} \in R$ , for  $\mathfrak{t} \in \text{Std}(\lambda)$ , are determined by the expression (2.6). The basis  $\mathcal{M}$  is called *Murphy basis* for  $H_{2k+1,n}$  and the basis (2.7) is referred to as the *Murphy basis* for  $S^{\lambda}$ . Notice that the  $H_{2k+1,n}$ -module  $S^{\lambda}$  is dual to Specht module in [4].

Applying the general theory of cellular algebra, the bilinear form on  $S^{\lambda}$  is the unique symmetric  $R$ -bilinear map from  $S^{\lambda} \times S^{\lambda}$  to  $R$  such that

$$(2.9) \quad \langle c_{\mathfrak{s}}, c_{\mathfrak{t}} \rangle c_{\lambda} \equiv c_{\mathfrak{s}}c_{\mathfrak{t}}^* \pmod{\check{\mathcal{H}}_{2k+1,n}^{\lambda}}$$

for all  $\mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda)$ . Then,  $\text{rad } S^{\lambda} = \{x \in S^{\lambda} \mid \langle x, y \rangle = 0 \text{ for all } y \in S^{\lambda}\}$  is a  $H_{2k+1,n}$ -submodule of  $S^{\lambda}$ . For each partition  $\lambda$  of  $n - 2k$ , denote  $D^{\lambda} = S^{\lambda} / \text{rad } S^{\lambda}$  a right  $H_{2k+1,n}$ -module.

Let  $e(q^2)$  be the least positive integer  $m$  such that  $[m]_{q^2} = 1 + q^2 + q^4 + \dots + q^{2(m-1)} = 0$  if that exists, and let  $e(q^2) = \infty$  otherwise. Recall that a partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_f)$  of  $n - 2k$  is  $e(q^2)$ -restricted if  $\lambda_i - \lambda_{i+1} < e(q^2)$  for all  $i \geq 1$ .

For partitions  $\lambda, \mu$  of  $n - 2k$  and  $D^{\mu} \neq 0$ , let  $d_{\lambda\mu} = [S^{\lambda} : D^{\mu}]$  be the composition multiplicity of  $D^{\mu}$  in  $S^{\lambda}$ . The following classification of the simple  $H_{2k+1,n}$ -modules is given by Dipper and James (see [4], Theorem 7.6 or [17], Theorem 3.43).

**Theorem 2.2.** *Suppose that  $R$  is a field.*

- (1)  $\{D^{\mu} \mid \mu \text{ an } e(q^2)\text{-restricted partition of } n - 2k\}$  is a complete set of non-isomorphic simple  $H_{2k+1,n}$ -modules.
- (2) Suppose that  $\mu$  is an  $e(q^2)$ -restricted partition of  $n - 2k$  and that  $\lambda$  is a partition of  $n - 2k$ . Then  $d_{\mu\mu} = 1$  and  $d_{\lambda\mu} \neq 0$  only if  $\lambda \supseteq \mu$ .

**Corollary 2.3.** ([17], Corollary 3.44) *Suppose that  $R$  is a field. Then the following statements are equivalent.*

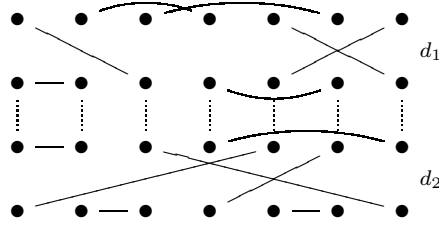
- (1)  $H_{2k+1,n}$  is (split) semisimple;
- (2)  $S^{\lambda} = D^{\lambda}$  for all partitions  $\lambda$  of  $n - 2k$ ;
- (3)  $e(q^2) > n - 2k$ .

**2.3. The Brauer algebra.** Brauer algebras were introduced first by Richard Brauer [2] in order to study the  $n$ th tensor power of the defining representation of the orthogonal and symplectic groups. Afterwards, they were studied in more detail by various mathematicians. We refer the reader to work of Hanlon and Wales ([10, 11]), Doran,

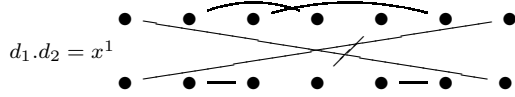
Wales and Hanlon [6], Graham and Lehrer [9] or Koenig and Xi [15], Wenzl [21] for more information.

The Brauer algebra is defined over the ring  $\mathbb{Z}[x]$  by a basis given by diagrams with  $2n$  vertices, arranged in two rows, and  $n$  edges, where each vertex belongs to exactly one edge. The edges which connect two vertices on the same row are called *horizontal edges*. The other ones are called *vertical edges*. We denote by  $D_n(x)$  Brauer algebra. The vertices of diagrams are numbered 1 to  $n$  from left to right in both the top and the bottom. The multiplication of two basis diagrams  $d_1$  and  $d_2$  is a concatenation in the following way: We put diagram  $d_1$  on top of  $d_2$  such that all vertices in the bottom row of  $d_1$  coincide with all upper vertices of  $d_2$ . Now draw an edge from vertex  $i$  in the bottom row of  $d_1$  to vertex  $i$  in top row of  $d_2$  for all  $i$ . The resulting diagram consists of parts that start and finish in top row of  $d_1$  and bottom row of  $d_2$  respectively, as well as some cycles that use only vertices in the middle two rows. Let  $\gamma(d_1, d_2)$  denote the number of these internal cycles. The product  $d_1 \cdot d_2$  in  $D_n(x)$  is then defined to be this resulting diagram without internal cycles, multiplied by  $x$  taken to the power  $\gamma(d_1, d_2)$ . Here  $x$  is a variable.

**Example 2.4.** Let us consider in  $D_7(x)$  the product of  $d_1$  and  $d_2$



and the resulting diagram is



In ([2], Section 5) R. Brauer points out that each basis diagram on  $D_n(x)$  which has exactly  $2k$  horizontal edges can be obtained in the form  $\omega_1 e_{(k)} \omega_2$  where  $\omega_1$  and  $\omega_2$  are permutations in  $S_n$ , and  $e_{(k)}$  is the following diagram:

$$(2.10) \quad \begin{array}{ccccccc} \bullet & - & \bullet & \dots & \bullet & - & \bullet & \bullet & \bullet & \dots & \bullet \\ \bullet & - & \bullet & \dots & \bullet & - & \bullet & \bullet & \bullet & \dots & \bullet \end{array},$$

where each row has exactly  $k$  horizontal edges.

**2.3.1. Length function for Brauer algebra  $D_n(N)$ .** Generalizing the length of elements in reflection groups, Wenzl [22] defined a length function for a basis diagram of  $D_n(N)$  as follows

For a basis diagram  $d \in D_n(N)$  with exactly  $2k$  horizontal edges, the definition of the length  $\ell(d)$  is given by

$$\ell(d) = \min\{\ell(\omega_1) + \ell(\omega_2) \mid \omega_1 e_{(k)} \omega_2 = d, \omega_1, \omega_2 \in S_n\}.$$

Recall that here we see a permutation  $\omega$  of a symmetric group as a diagram of the Brauer algebra with no horizontal edge. The product  $\omega_1 \omega_2$  is a concatenation of two diagram  $\omega_1$  and  $\omega_2$ .

As indicated in [22], a permutation  $\omega \in S_n$  can be written uniquely in the form  $\omega = t_1 \dots t_{n-2} t_{n-1}$ , where  $t_j = 1$  or  $t_j = s_j s_{j-1} s_{j-2} \dots s_{i_j} =: s_{j,i_j}$  with  $1 \leq i_j \leq j < n$ . Denote  $B_k$  the set of all elements of the form  $t_2 t_4 \dots t_{2k-2} t_{2k} t_{2k+1} \dots t_{n-2} t_{n-1}$ .

For  $k$  an integer,  $0 \leq k \leq [n/2]$ , let  $\mathcal{D}_{k,n}$  be the set of all diagrams  $d$  in which: A diagram has exactly  $k$  horizontal edges on each row, its top row is like a row of the diagram  $e_{(k)}$ , and there is no crossing between any two vertical edges. Set

$$(2.11) \quad B_{k,n} = \{\omega \in B_k \mid \ell(d) = \ell(\omega) \text{ with } d = e_{(k)}\omega \in \mathcal{D}_{k,n}\}.$$

This definition is going to be used in the following section on the  $q$ -Brauer algebra. For more detail we refer the reader to Section 3.3 in [7].

### 3. A MURPHY BASIS OF THE $q$ -BRAUER ALGEBRA

**3.1. The  $q$ -Brauer algebra.** From now on, we abbreviate  $H_n$  replacing  $\mathcal{H}_n(q^2)$ . The generic  $q$ -Brauer algebra, which contains the Hecke algebra of the symmetric group  $H_n$  as a subalgebra, is defined below.

**Definition 3.1.** Let  $r$  and  $q$  be invertible elements over the ring  $\mathbb{Z}[q^{\pm 1}, r^{\pm 1}, (\frac{r-r^{-1}}{q-q^{-1}})^{\pm 1}]$ . Moreover, if  $q = 1$ , then assume that  $r = q^N$  with  $N \in \mathbb{Z} \setminus \{0\}$ . The  $q$ -Brauer algebra  $Br_n(r^2, q^2)$  over  $\mathbb{Z}[q^{\pm 1}, r^{\pm 1}, (\frac{r-r^{-1}}{q-q^{-1}})^{\pm 1}]$  is the algebra defined via generators  $g_1, g_2, g_3, \dots, g_{n-1}$  and  $e$  and relations

(H) The elements  $g_1, g_2, g_3, \dots, g_{n-1}$  satisfy the relations of the Hecke algebra  $H_n$ ;

$$(E_1) \quad e^2 = \frac{r-r^{-1}}{q-q^{-1}}e;$$

$$(E_2) \quad eg_i = g_i e \text{ for } i > 2, eg_1 = g_1 e = q^2 e, eg_2 e = r q e \text{ and } eg_2^{-1} e = (r q)^{-1} e;$$

$$(E_3) \quad g_2 g_3 g_1^{-1} g_2^{-1} e_{(2)} = e_{(2)} g_2 g_3 g_1^{-1} g_2^{-1}, \text{ where } e_{(2)} = e(g_2 g_3 g_1^{-1} g_2^{-1})e.$$

Let

$$g_{l,m}^+ = \begin{cases} g_l g_{l+1} \dots g_m & \text{if } l \leq m; \\ g_l g_{l-1} \dots g_m & \text{if } l > m, \end{cases}$$

and

$$g_{l,m}^- = \begin{cases} g_l^{-1} g_{l+1}^{-1} \dots g_m^{-1} & \text{if } l \leq m; \\ g_l^{-1} g_{l-1}^{-1} \dots g_m^{-1} & \text{if } l > m, \end{cases}$$

for  $1 \leq l, m \leq n$ .

Let  $k$  be an integer,  $1 \leq k \leq [n/2]$ . The elements  $e_{(k)}$  in  $Br_n(r^2, q^2)$  are defined inductively by  $e_{(1)} = e$  and by

$$(3.1) \quad e_{(k+1)} = e g_{2,2k+1}^+ g_{1,2k}^- e_{(k)}.$$

**Remark 3.2.** 1. We keep the notation  $e_{(k)}$  as used in [22] and [7], which describes both the Brauer diagram  $e_{(k)}$  (see (2.10)) of the Brauer algebra and the element  $e_{(k)}$  (in (3.1)) of the  $q$ -Brauer algebra.

2. The definition of the  $q$ -Brauer algebra above is a generic version of the one introduced by Wenzl (see [24], Definition 2.1). This means when  $r = q^N$ , these definitions are the same. The algebra defined above is also isomorphic to another version of  $q$ -Brauer algebra,  $Br_n(r, q)$ , used by Dung in [7]. In fact, it can be obtained by in  $Br_n(r, q)$  we

substitute *old*  $q$ ,  $r$  and  $e$  by  $q^2$ ,  $r^2$ , and  $(q^{-1}r)e$ , respectively. This implies that the  $q$ -Brauer  $Br_n(r^2, q^2)$  has similar properties as those of  $Br_n(r, q)$ .

3. To relate the  $q$ -Brauer algebra to the Brauer algebra over a field of any characteristic, we need another version of the  $q$ -Brauer algebra. The definition is the following:

Fix  $N \in \mathbb{Z} \setminus \{0\}$  and let  $[N] = 1 + q^2 + \dots + q^{2(N-1)}$ , where  $q$  is an invertible element in an arbitrary commutative noetherian ring  $R$  containing  $\mathbb{Z}[q^{\pm 1}, r^{\pm 1}, [N]^{\pm 1}]$ . The  $q$ -Brauer algebra  $Br_n(N)$  is an algebra over  $R$  defined by generators  $g_1, g_2, \dots, g_{n-1}$  and  $e$  and relations  $(H)$ ,  $(E_3)$  as before and

$$(E'_1) \quad e^2 = [N]e;$$

$$(E'_2) \quad eg_i = g_i e \text{ for } i > 2, \quad eg_1 = g_1 e = q^2 e, \quad eg_2 e = q^{N+1} e \text{ and } eg_2^{-1} e = (q)^{-1-N} e.$$

It is clear that in the case  $q = 1$  the  $q$ -Brauer algebra  $Br_n(N)$  coincides with the Brauer algebra  $D_n(N)$ . Notice that the other versions of the  $q$ -Brauer algebra recover the Brauer algebra over fields allowing to form the limit  $q \rightarrow 1$ , such as the field of real or complex numbers (see Remark 3.1(1) in [22] for more detail).

4. Both  $Br_n(r^2, q^2)$  and  $Br_n(N)$  have an  $R$ -linear involution  $*$  defined by  $e^* = e$ ,  $g_i^* = g_i$ ,  $1 \leq i \leq n-1$ . This involution is the same as the involution of  $Br_n(r, q)$  shown to exist in Proposition 3.12 [7], and it is compatible with the involution of the Hecke algebra  $H_n$  defined in Section 2.2.

In this article, the proofs for both  $Br_n(r^2, q^2)$  and  $Br_n(N)$  are the same. We will only give them for one version, sometimes without explicitly mentioning the other version.

5. Let  $k$  be an integer,  $0 \leq k \leq [n/2]$ . By Definition (3.1) it is straightforward to check that the element  $e_{(k)}$  commutes with  $g_\omega$  for  $\omega \in H_{2k+1, n}$ , that is,  $e_{(k)}g_\omega = g_\omega e_{(k)}$ .

Recall from [7] Section 4.1 that if  $k$  is an integer,  $0 \leq k \leq [n/2]$ ,  $J_n(k)$  is the  $R$ -module generated by the basis elements  $g_d$  of the  $q$ -Brauer algebra, where  $d$  is a Brauer diagram whose number of vertical edges are less than or equal  $n - 2k$ . Then  $J_n(k)$  is an ideal of the  $q$ -Brauer algebra and

$$(3.2) \quad J_n(k) = \sum_{j=k}^{[n/2]} H_n e_{(j)} H_n.$$

In the following we collect some results which are similar to those of  $Br_n(r, q)$  in [7]. Their proofs are the same as those of  $Br_n(r, q)$ .

**Lemma 3.3.** *The following statements hold for the  $q$ -Brauer algebra  $Br_n(r^2, q^2)$ .*

- (1)  $g_{2j+1}e_{(k)} = e_{(k)}g_{2j+1} = q^2 e_{(k)}$ , and  $g_{2j+1}^{-1}e_{(k)} = e_{(k)}g_{2j+1}^{-1} = q^{-2}e_{(k)}$  for  $0 \leq j < k$ ;
- (2)  $e_{(j)}e_{(k)} = e_{(k)}e_{(j)} = \left(\frac{r - r^{-1}}{q - q^{-1}}\right)^j e_{(k)}$  for any  $j \leq k$ ;
- (3)  $g_{2i-1, 2j}^+ e_{(k)} = g_{2j+1, 2i}^+ e_{(k)}$  and  $g_{2i-1, 2j}^- e_{(k)} = g_{2j+1, 2i}^- e_{(k)}$  for  $1 \leq i \leq j < k$ ;
- (4)  $e_{(k)}g_{2l, 1}^+ = e_{(k)}g_{2, 2l+1}^+$  and  $e_{(k)}g_{2l, 1}^- = e_{(k)}g_{2, 2l+1}^-$  for  $l < k$ ;
- (5)  $e_{(k)}g_{2j}g_{2j-1} = e_{(k)}g_{2j}g_{2j+1}$  and  $e_{(k)}g_{2j}^{-1}g_{2j-1}^{-1} = e_{(k)}g_{2j}^{-1}g_{2j+1}^{-1}$  for  $1 \leq j < k$ ;
- (6)  $e_{(k)}g_{2j, 2i-1}^+ = e_{(k)}g_{2i, 2j+1}^+$  and  $e_{(k)}g_{2j, 2i-1}^- = e_{(k)}g_{2i, 2j+1}^-$  for  $1 \leq i \leq j < k$ ;
- (7)  $e_{(k)}g_{2k, 2j-1}^- g_{2k+1, 2j}^+ e_{(j)} = \left(\frac{r - r^{-1}}{q - q^{-1}}\right)^{j-1} e_{(k+1)}$  for  $1 \leq j < k$ ;
- (8)  $e_{(k)}g_{2j}e_{(j)} = rq\left(\frac{r - r^{-1}}{q - q^{-1}}\right)^{j-1} e_{(k)}$  for  $1 \leq j \leq k$ ;
- (9)  $e_{(k)}H_n e_{(j)} \subset e_{(k)}H_{2j+1, n} + \sum_{m \geq k+1} H_n e_{(m)} H_n$ , where  $j \leq k$ ;



$$(10) \quad e_{(k+1)} = e_{(k)} g_{2k,1}^- g_{2k+1,2}^+ e.$$

**Lemma 3.4.** ([7], Lemma 4.10) *Let  $k, l$  be integers,  $0 < k \leq l \leq [n/2]$ , and let  $u$  be a permutation in  $B_{l,n}$  and  $\pi$  a permutation in  $S_{2l+1,n}$ . Then there exist  $a_{(\omega,u)} \in R$ , for  $v \in B_{k,n}$  and  $\omega \in S_{2k+1,n}$ , such that*

$$e_{(k)} g_{\pi} g_u = \sum_{\substack{\omega \in S_{2k+1,n} \\ v \in B_{k,n}}} a_{(\omega,v)} e_{(k)} g_{\omega} g_v.$$

The statement below is a special case of Proposition 4.12 in [7].

**Lemma 3.5.** *Let  $k$  be an integer,  $0 < k \leq [n/2]$ . If  $b \in Br_n(r^2, q^2)$ ,  $u \in B_{k,n}$ , then there exist  $a_{(\omega,v)} \in R$ , for  $\omega \in S_{2k+1,n}$  and  $v \in B_{k,n}$ , such that*

$$e_{(k)} g_u b \equiv \sum_{\substack{\omega \in S_{2k+1,n} \\ v \in B_{k,n}}} a_{(\omega,v)} e_{(k)} g_{\omega} g_v \pmod{J_n(k+1)}.$$

**Theorem 3.6.** ([7], Theorem 4.17) *Suppose that  $\Lambda$  is a commutative noetherian ring which contains  $R$  as a subring with the same identity. If  $q, r$  and  $\frac{r - r^{-1}}{q - q^{-1}}$  (resp.  $[N]$ ) are invertible in  $\Lambda$ , then the  $q$ -Brauer algebra  $Br_n(r^2, q^2)$  (resp.  $Br_n(N)$ ) over the ring  $\Lambda$  is cellular.*

The following statement gives an explicit cellular basis for the  $q$ -Brauer algebra  $Br_n(r^2, q^2)$ . The proof follows from Theorem 3.13, Propositions 3.14 and 4.12 in [7].

**Theorem 3.7.** *The  $q$ -Brauer algebra  $Br_n(r^2, q^2)$  (resp.  $Br_n(N)$ ) is freely generated as an  $R$ -module by the basis*

$$\{ g_u^* e_{(k)} g_{\pi} g_v \mid u, v \in B_{k,n} \text{ and } \pi \in S_{2k+1,n} \text{ for } 0 \leq k \leq [n/2] \}.$$

Moreover, the following statements hold.

(1) *The involution  $*$  satisfies*

$$* : g_u^* g_{\pi} e_{(k)} g_v \mapsto g_v^* g_{\pi}^* e_{(k)} g_u$$

*for all  $u, v \in B_{k,n}$  and  $\pi \in S_{2k+1,n}$ .*

(2) *Suppose that  $b \in Br_n(r^2, q^2)$  and let  $k$  be an integer,  $0 \leq k \leq [n/2]$ .*

*If  $u, v \in B_{k,n}$  and  $\pi \in S_{2k+1,n}$ , then there exist  $v_1 \in B_{k,n}$  and  $\pi_1 \in S_{2k+1,n}$  such that*

$$(3.3) \quad g_u^* e_{(k)} g_{\pi} g_v b \equiv \sum_{\substack{\pi_1 \in S_{2k+1,n} \\ v_1 \in B_{k,n}}} a_{(\pi_1, v_1)} g_u^* e_{(k)} g_{\pi_1} g_{v_1} \pmod{J_n(k+1)}.$$

For  $k$  an integer,  $0 \leq k \leq [n/2]$ , the  $R$ -module  $J_n(k+1)$  has a basis

$$(3.4) \quad \{ g_u^* e_{(l)} g_{\pi} g_v \mid u, v \in B_{l,n} \text{ and } \pi \in S_{2l+1,n} \text{ for all } k < l \leq [n/2] \}.$$

**3.2. Main theorem.** For  $k$  an integer,  $0 \leq k \leq [n/2]$ , let

$$\Lambda_n := \{(k, \lambda) \mid \text{for all } 0 \leq k \leq [n/2], \text{ and } \lambda \text{ is a partition of } n - 2k\}.$$

For  $(k, \mu) \in \Lambda_n$ , define the element

$$(3.5) \quad m_\mu = e_{(k)} c_\mu \quad \text{where } c_\mu \text{ is defined in (2.3).}$$

**Example 3.8.** Let  $n = 10$  and  $\mu = (3, 2, 1)$ . The example in (2.2) yields the subgroup  $S_\mu = \langle s_5, s_6, s_8 \rangle$  and  $m_\mu = e_{(2)} \sum_{\sigma \in S_\mu} g_\sigma = e_{(2)}(1 + g_5)(1 + g_6 + g_6 g_5)(1 + g_8)$ .

For  $(k, \lambda) \in \Lambda_n$ , define  $\mathcal{I}_n(k, \lambda)$  to be the set of ordered pairs

$$(3.6) \quad \mathcal{I}_n(k, \lambda) = \text{Std}(\lambda) \times B_{k,n} = \{(\mathfrak{s}, u) : \mathfrak{s} \in \text{Std}(\lambda) \text{ and } u \in B_{k,n}\}.$$

Let  $\check{B}r_n^\lambda$  be the  $R$ -module with spanning set

$$(3.7) \quad \left\{ x_{(\mathfrak{s}, u)(\mathfrak{t}, v)}^\mu := g_u^* g_{d(\mathfrak{s})}^* m_\mu g_{d(\mathfrak{t})} g_v \mid \begin{array}{l} (\mathfrak{s}, u), (\mathfrak{t}, v) \in \mathcal{I}_n(l, \mu) \\ \mu \triangleright \lambda \text{ for } (l, \mu), (k, \lambda) \in \Lambda_n \end{array} \right\}.$$

**Lemma 3.9.** Suppose that  $(k, \lambda) \in \Lambda_n$ , then  $J_n(k+1) \subseteq \check{B}r_n^\lambda$  and  $\check{B}r_n^\lambda$  is an ideal of the  $q$ -Brauer algebra.

*Proof.* By (3.4), every basis element in  $J_n(k+1)$  is of the form  $g_u^* e_{(l)} g_\pi g_v$  where  $u, v \in B_{l,n}$  and  $\pi \in S_{2l+1,n}$ ,  $k+1 \leq l \leq [n/2]$ . By the definition in Theorem 2.1,  $g_\pi = g_{d(\mathfrak{s})}^* c_\mu g_{d(\mathfrak{t})}$  with  $(l, \mu) \in \Lambda_n$ . Since  $n - 2l < n - 2k$ , the definition of dominance order in Section 2.1 implies that  $\mu \triangleright \lambda$ . Thus,  $g_u^* e_{(l)} g_\pi g_v = g_u^* g_{d(\mathfrak{s})}^* m_\mu g_{d(\mathfrak{t})} g_v = x_{(\mathfrak{s}, u)(\mathfrak{t}, v)}^\mu \in \check{B}r_n^\lambda$ , meaning  $J_n(k+1) \subseteq \check{B}r_n^\lambda$ .

To prove the second statement, it is sufficient to show that  $x_{(\mathfrak{s}, u)(\mathfrak{t}, v)}^\mu \cdot b \in \check{B}r_n^\lambda$ , where  $x_{(\mathfrak{s}, u)(\mathfrak{t}, v)}^\mu \in \check{B}r_n^\lambda$  (with  $\mu \triangleright \lambda$ ) and  $b$  is a basis element of the  $q$ -Brauer algebra  $Br_n(r^2, q^2)$ . Lemma 3.5 implies

$$(e_{(l)} g_v) b \equiv \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} a_{(\pi_1, v_1)} e_{(l)} g_{\pi_1} g_{v_1} \pmod{J_n(l+1)}.$$

Notice that in (2.5) of Theorem 2.1  $c_{\mathbf{1t}} = \mathbf{1}c_\mu g_{d(\mathbf{t})}$ . We have the calculation:

$$\begin{aligned}
x_{(\mathfrak{s},u)(\mathbf{t},v)}^\mu \cdot b &= (g_u^* g_{d(\mathfrak{s})}^* m_\mu g_{d(\mathbf{t})} g_v) b = (g_u^* g_{d(\mathfrak{s})}^* c_\mu g_{d(\mathbf{t})}) (e_{(l)} g_v b) \\
&= \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} a_{(\pi_1, v_1)} g_u^* g_{d(\mathfrak{s})}^* e_{(l)} (\mathbf{1} c_\mu g_{d(\mathbf{t})} g_{\pi_1}) g_{v_1} + J_n(l+1) \\
&\stackrel{(2.5)}{=} \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} a_{(\pi_1, v_1)} g_u^* g_{d(\mathfrak{s})}^* e_{(l)} (c_{\mathbf{1t}} g_{\pi_1}) g_{v_1} + J_n(l+1) \\
&\stackrel{(2.6)}{=} \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} a_{(\pi_1, v_1)} g_u^* g_{d(\mathfrak{s})}^* e_{(l)} \left( \sum_{\mathbf{t}_1 \in \text{Std}(\mu)} a_{\mathbf{t}_1} c_{\mathbf{1t}_1} + \check{\mathcal{H}}_{2l+1,n}^\mu \right) g_{v_1} + J_n(l+1) \\
&\stackrel{(2.5)}{=} \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} a_{(\pi_1, v_1)} g_u^* g_{d(\mathfrak{s})}^* e_{(l)} \left( \sum_{\mathbf{t}_1 \in \text{Std}(\mu)} a_{\mathbf{t}_1} c_\mu g_{d(\mathbf{t}_1)} + \check{\mathcal{H}}_{2l+1,n}^\mu \right) g_{v_1} + J_n(l+1) \\
&= \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} \sum_{\mathbf{t}_1 \in \text{Std}(\mu)} a_{\mathbf{t}_1} a_{(\pi_1, v_1)} (g_u^* g_{d(\mathfrak{s})}^* e_{(l)} c_\mu g_{d(\mathbf{t}_1)} g_{v_1}) \\
&+ \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} a_{(\pi_1, v_1)} (g_u^* g_{d(\mathfrak{s})}^* e_{(l)} \check{\mathcal{H}}_{2l+1,n}^\mu g_{v_1}) + J_n(l+1) \\
&\stackrel{(3.5), (2.4)}{=} \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} \sum_{\mathbf{t}_1 \in \text{Std}(\mu)} a_{\mathbf{t}_1} a_{(\pi_1, v_1)} (g_u^* g_{d(\mathfrak{s})}^* m_\mu g_{d(\mathbf{t}_1)} g_{v_1}) \\
&+ \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} \sum_{\substack{\mu_2 \vdash n-2l, \mu_2 \triangleright \mu \\ \mathfrak{s}_2, \mathbf{t}_2 \in \text{Std}(\mu_2)}} a_{(\pi_1, v_1)} a_{(\mathfrak{s}_2, \mathbf{t}_2)} (g_u^* g_{d(\mathfrak{s}_2)}^* m_{\mu_2} g_{d(\mathbf{t}_2)} g_{v_1}) + J_n(l+1) \\
&= \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} \sum_{\mathbf{t}_1 \in \text{Std}(\mu)} a_{\mathbf{t}_1} a_{(\pi_1, v_1)} (x_{(\mathfrak{s},u)(\mathbf{t}_1, v_1)}^\mu) \\
&+ \sum_{\substack{\pi_1 \in S_{2l+1,n} \\ v_1 \in B_{l,n}}} \sum_{\substack{\mu_2 \vdash n-2l, \mu_2 \triangleright \mu \\ \mathfrak{s}_2, \mathbf{t}_2 \in \text{Std}(\mu_2)}} a_{(\pi_1, v_1)} a_{(\mathfrak{s}_2, \mathbf{t}_2)} (x_{(\mathfrak{s}_2, u)(\mathbf{t}_2, v_1)}^{\mu_2}) + J_n(l+1),
\end{aligned}$$

where  $a_{(\pi_1, v_1)}$ ,  $a_{\mathbf{t}_1}$  and  $a_{\mathfrak{s}_2, \mathbf{t}_2}$  are in  $R$ , and  $\check{\mathcal{H}}_{2l+1,n}^\mu$  is the ideal of  $H_{2l+1,n}$  defined in (2.4). By Lemma 3.9 and the assumption  $\mu \triangleright \lambda$  (hence  $l \geq k$ ), all elements occurring in the last formula are in  $\check{B}r_n^\lambda$ , and hence, so is  $x_{(\mathfrak{s},u)(\mathbf{t},v)}^\mu \cdot b$ .  $\square$

The main result of this section is the following.

**Theorem 3.10.** *The  $q$ -Brauer algebra  $Br_n(r^2, q^2)$  (resp.  $Br_n(N)$ ) is freely generated as an  $R$ -module by the collection*

$$(3.8) \quad \left\{ x_{(\mathfrak{s},u)(\mathbf{t},v)}^\lambda = g_u^* g_{d(\mathfrak{s})}^* m_\lambda g_{d(\mathbf{t})} g_v \mid (\mathfrak{s}, u), (\mathbf{t}, v) \in \mathcal{I}_n(k, \lambda), \text{ for } (k, \lambda) \in \Lambda_n \right\}.$$

Moreover, the following statements hold.

- (1) The involution  $*$  sends  $x_{(\mathfrak{s},u)(\mathbf{t},v)}^\lambda$  to  $(x_{(\mathfrak{s},u)(\mathbf{t},v)}^\lambda)^* = x_{(\mathbf{t},v)(\mathfrak{s},u)}^\lambda$  for all  $(\mathbf{t}, v), (\mathfrak{s}, u) \in \mathcal{I}_n(k, \lambda)$ .

(2) Suppose that  $b \in Br_n(r^2, q^2)$ ,  $(k, \lambda) \in \Lambda_n$  and  $(\mathfrak{s}, u), (\mathfrak{t}, v) \in \mathcal{I}_n(k, \lambda)$ . Then, there exist  $a_{(\mathfrak{t}_2, v_2)} \in R$ ,  $(\mathfrak{t}_2, v_2) \in \mathcal{I}_n(k, \lambda)$  such that

$$(3.9) \quad x_{(\mathfrak{s}, u)(\mathfrak{t}, v)}^\lambda \cdot b \equiv \sum_{(\mathfrak{t}_2, v_2) \in \mathcal{I}_n(k, \lambda)} a_{(\mathfrak{t}_2, v_2)} x_{(\mathfrak{s}, u)(\mathfrak{t}_2, v_2)}^\lambda \pmod{\check{B}r_n^\lambda}.$$

*Proof.* Let  $b$  be an arbitrary element in  $Br_n(r^2, q^2)$ . Then by Theorem 3.7,  $b$  can be expressed as an  $R$ -linear combination

$$(3.10) \quad b = \sum_j a_j g_{u_j}^* g_{\pi_j} e_{(k_j)} g_{v_j},$$

where  $(k_j, \lambda) \in \Lambda_n$ ,  $a_j \in R$ ,  $\pi_j \in S_{2k+1, n}$ , and  $u_j, v_j \in B_{k_j, n}$ . Using the definition in Theorem 2.1 the element  $g_{\pi_j}$  has an expression

$$g_{\pi_j} = g_{d(\mathfrak{s}_j)}^* c_{\lambda_j} g_{d(\mathfrak{t}_j)}$$

with  $\mathfrak{s}_j, \mathfrak{t}_j \in Std(\lambda_j)$ . Replace  $g_{\pi_j}$  in (3.10) by the last equation,

$$\begin{aligned} b &= \sum_j a_j g_{u_j}^* g_{d(\mathfrak{s}_j)}^* c_{\lambda_j} g_{d(\mathfrak{t}_j)} e_{(k_j)} g_{v_j} = \sum_j a_j g_{u_j}^* g_{d(\mathfrak{s}_j)}^* e_{(k_j)} c_{\lambda_j} g_{d(\mathfrak{t}_j)} g_{v_j} \\ &\stackrel{(3.5)}{=} \sum_j a_j g_{u_j}^* g_{d(\mathfrak{s}_j)}^* m_{\lambda_j} g_{d(\mathfrak{t}_j)} g_{v_j} = \sum_j a_j x_{(\mathfrak{s}_j, u_j)(\mathfrak{t}_j, v_j)}^\lambda, \end{aligned}$$

where  $(\mathfrak{s}_j, u_j), (\mathfrak{t}_j, v_j) \in \mathcal{I}_n(k_j, \lambda_j)$ .

Therefore, the set (3.8) linearly spans  $Br_n(r^2, q^2)$ . The independence of (3.8) follows from the linear independences in Theorems 2.1 and 3.7.

The statement (1) is obtained by combining the involution  $*$  on the Hecke algebra  $H_n$  and Lemma 3.3(10) in which  $(e_{(k)})^* = e_{(k)}$ .

The statement (2) is shown as follows: Let  $(k, \lambda), (l, \mu) \in \Lambda_n$ . As the same arguments in the proof above, it suffices to consider the product  $x_{(\mathfrak{s}, u)(\mathfrak{t}, v)}^\lambda \cdot b$ , where  $b = x_{(\mathfrak{s}_1, u_1)(\mathfrak{t}_1, v_1)}^\mu$  with  $(\mathfrak{s}_1, u_1), (\mathfrak{t}_1, v_1) \in \mathcal{I}_n(l, \mu)$  is a basis element in  $Br_n(r^2, q^2)$ . Subsequently, consider two cases with respect to partitions  $\mu$  and  $\lambda$ .

The first case is  $\mu \triangleright \lambda$ : Then the definition of the dominance order implies that  $k \leq l$ . So, by Lemma 3.4 [23], we get

$$e_{(k)} g_v g_{u_1}^* e_{(l)} \in H_{2k+1, n} e_{(l)} + \sum_{m \geq l+1} H_n e_{(m)} H_n.$$

Hence,

$$\begin{aligned} (c_\lambda g_{d(\mathfrak{t})})(e_{(k)} g_v g_{u_1}^* e_{(l)})(g_{d(\mathfrak{s}_1)}^* c_\mu) &\in c_\lambda g_{d(\mathfrak{t})} H_{2k+1, n} e_{(l)} g_{d(\mathfrak{s}_1)}^* c_\mu + \sum_{m \geq l+1} H_n e_{(m)} H_n \\ &\stackrel{(3.5)}{\subseteq} H_{2k+1, n} g_{d(\mathfrak{s}_1)}^* m_\mu + \sum_{m \geq l+1} H_n e_{(m)} H_n \stackrel{k \leq l}{\subseteq} H_{2k+1, n} g_{d(\mathfrak{s}_1)}^* m_\mu + \sum_{m \geq k+1} H_n e_{(m)} H_n \\ &\stackrel{(3.2)}{\subseteq} H_{2k+1, n} g_{d(\mathfrak{s}_1)}^* m_\mu + J_n(k+1) \subseteq \check{B}r_n^\lambda. \end{aligned}$$

Since  $(c_\lambda g_{d(\mathfrak{t})})(e_{(k)} g_v g_{u_1}^* e_{(l)})(g_{d(\mathfrak{s}_1)}^* c_\mu) \stackrel{(3.5)}{=} (m_\lambda g_{d(\mathfrak{t})} g_v)(g_{u_1}^* g_{d(\mathfrak{s}_1)}^* m_\mu)$ , the last inclusion yields

$$(x_{(\mathfrak{s}, u)(\mathfrak{t}, v)}^\lambda)(x_{(\mathfrak{s}_1, u_1)(\mathfrak{t}_1, v_1)}^\mu) = (g_u^* g_{d(\mathfrak{s})}^* m_\lambda g_{d(\mathfrak{t})} g_v)(g_{u_1}^* g_{d(\mathfrak{s}_1)}^* m_\mu g_{d(\mathfrak{t}_1)} g_{v_1}) \in g_u^* g_{d(\mathfrak{s})}^* \check{B}r_n^\lambda g_{d(\mathfrak{t}_1)} g_{v_1} \subseteq \check{B}r_n^\lambda.$$

Thus,  $x_{(\mathfrak{s},u)(\mathfrak{t},v)}^\lambda(x_{(\mathfrak{s}_1,u_1)(\mathfrak{t}_1,v_1)}^\mu) \equiv 0 \pmod{\check{B}r_n^\lambda}$ , namely,

$$x_{(\mathfrak{s},u)(\mathfrak{t},v)}^\lambda \cdot b \equiv 0 \pmod{\check{B}r_n^\lambda}.$$

The second case is  $\lambda \supseteq \mu$ , that is  $l \leq k$ : Using Lemma 3.3(9), it yields

$$e_{(k)}g_vg_{u_1}^*e_{(l)} \in e_{(k)}H_{2l+1,n} + \sum_{m \geq k+1} H_n e_{(m)} H_n.$$

Applying Lemma 3.3(9) and the same arguments as in the previous case implies that

$$\begin{aligned} (c_\lambda g_{d(\mathfrak{t})})(e_{(k)}g_vg_{u_1}^*e_{(l)})(g_{d(\mathfrak{s}_1)}^*c_\mu) &= (m_\lambda g_{d(\mathfrak{t})}g_v)(g_{u_1}^*g_{d(\mathfrak{s}_1)}^*m_\mu) \\ &\in c_\lambda g_{d(\mathfrak{t})}e_{(k)}H_{2l+1,n}g_{d(\mathfrak{s}_1)}^*c_\mu + \sum_{m \geq k+1} H_n e_{(m)} H_n \\ &\stackrel{(3.2),(3.5)}{\subseteq} m_\lambda H_{2l+1,n} + J_n(k+1) \subseteq m_\lambda H_{2l+1,n} + \check{B}r_n^\lambda \quad (\text{by Lemma 3.9}). \end{aligned}$$

Hence,

$$\begin{aligned} x_{(\mathfrak{s},u)(\mathfrak{t},v)}^\lambda \cdot b &= (g_u^*g_{d(\mathfrak{s})}^*m_\lambda g_{d(\mathfrak{t})}g_v)(g_{u_1}^*g_{d(\mathfrak{s}_1)}^*m_\mu g_{d(\mathfrak{t}_1)}g_{v_1}) \\ &\in g_u^*g_{d(\mathfrak{s})}^*m_\lambda H_{2l+1,n}g_{d(\mathfrak{t}_1)}g_{v_1} + \check{B}r_n^\lambda \subseteq g_u^*g_{d(\mathfrak{s})}^*m_\lambda H_{2l+1,n}g_{v_1} + \check{B}r_n^\lambda. \end{aligned}$$

It implies that  $x_{(\mathfrak{s},u)(\mathfrak{t},v)}^\lambda \cdot b$  can be rewritten as an  $R$ -linear combination

$$\begin{aligned} x_{(\mathfrak{s},u)(\mathfrak{t},v)}^\lambda \cdot b &= g_u^*g_{d(\mathfrak{s})}^*m_\lambda \left( \sum_{\pi_1 \in S_{2l+1,n}} a_{\pi_1} g_{\pi_1} \right) g_{v_1} + \check{B}r_n^\lambda \\ &\stackrel{(3.5)}{=} g_u^*g_{d(\mathfrak{s})}^*c_\lambda \left( \sum_{\pi_1 \in S_{2l+1,n}} a_{\pi_1} e_{(k)} g_{\pi_1} g_{v_1} \right) + \check{B}r_n^\lambda \\ &\stackrel{L3.4}{=} g_u^*g_{d(\mathfrak{s})}^*c_\lambda \left( \sum_{\pi_1 \in S_{2l+1,n}} a_{\pi_1} \left( \sum_{\substack{\omega_{\pi_1} \in S_{2k+1,n} \\ v_{\pi_1} \in B_{k,n}}} a_{(\omega_{\pi_1}, v_{\pi_1})} e_{(k)} g_{\omega_{\pi_1}} g_{v_{\pi_1}} \right) \right) + \check{B}r_n^\lambda \\ &= \sum_{\pi_1 \in S_{2l+1,n}} a_{\pi_1} \left( \sum_{\substack{\omega_{\pi_1} \in S_{2k+1,n} \\ v_{\pi_1} \in B_{k,n}}} a_{(\omega_{\pi_1}, v_{\pi_1})} g_u^*g_{d(\mathfrak{s})}^*e_{(k)}(c_\lambda g_{\omega_{\pi_1}}) g_{v_{\pi_1}} \right) + \check{B}r_n^\lambda \\ &\stackrel{(2.6)}{=} \sum_{\pi_1 \in S_{2l+1,n}} a_{\pi_1} \left( \sum_{\substack{\omega_{\pi_1} \in S_{2k+1,n} \\ v_{\pi_1} \in B_{k,n}}} a_{(\omega_{\pi_1}, v_{\pi_1})} g_u^*g_{d(\mathfrak{s})}^*e_{(k)} \left( \sum_{\mathfrak{t}_{\omega_{\pi_1}} \in Std(\lambda)} a_{\mathfrak{t}_{\omega_{\pi_1}}} c_{1\mathfrak{t}_{\omega_{\pi_1}}} + \check{\mathcal{H}}_{2k+1,n}^\lambda \right) g_{v_{\pi_1}} \right) + \check{B}r_n^\lambda \\ &= \sum_{\pi_1 \in S_{2l+1,n}} a_{\pi_1} \left( \sum_{\substack{\omega_{\pi_1} \in S_{2k+1,n} \\ v_{\pi_1} \in B_{k,n}}} a_{(\omega_{\pi_1}, v_{\pi_1})} \left( \sum_{\mathfrak{t}_{\omega_{\pi_1}} \in Std(\lambda)} a_{\mathfrak{t}_{\omega_{\pi_1}}} g_u^*g_{d(\mathfrak{s})}^*e_{(k)} c_{1\mathfrak{t}_{\omega_{\pi_1}}} g_{v_{\pi_1}} \right) \right) \\ &\quad + \sum_{\pi_1 \in S_{2l+1,n}} a_{\pi_1} \left( \sum_{\substack{\omega_{\pi_1} \in S_{2k+1,n} \\ v_{\pi_1} \in B_{k,n}}} a_{(\omega_{\pi_1}, v_{\pi_1})} g_u^*g_{d(\mathfrak{s})}^*e_{(k)} \check{\mathcal{H}}_{2k+1,n}^\lambda g_{v_{\pi_1}} \right) + \check{B}r_n^\lambda \\ &= \sum_{\pi_1 \in S_{2l+1,n}} a_{\pi_1} \left( \sum_{\substack{\omega_{\pi_1} \in S_{2k+1,n} \\ v_{\pi_1} \in B_{k,n}}} a_{(\omega_{\pi_1}, v_{\pi_1})} \left( \sum_{\mathfrak{t}_{\omega_{\pi_1}} \in Std(\lambda)} a_{\mathfrak{t}_{\omega_{\pi_1}}} x_{(\mathfrak{s},u)(\mathfrak{t}_{\omega_{\pi_1}}, v_{\pi_1})}^\lambda \right) \right) \\ &\quad + \sum_{\pi_1 \in S_{2l+1,n}} a_{\pi_1} \left( \sum_{\substack{\omega_{\pi_1} \in S_{2k+1,n} \\ v_{\pi_1} \in B_{k,n}}} a_{(\omega_{\pi_1}, v_{\pi_1})} g_u^*g_{d(\mathfrak{s})}^*e_{(k)} \check{\mathcal{H}}_{2k+1,n}^\lambda g_{v_{\pi_1}} \right) + \check{B}r_n^\lambda, \end{aligned}$$

where  $a_{\pi_1}$ ,  $a_{(\omega_{\pi_1}, v_{\pi_1})}$ , and  $a_{t_{\omega_{\pi_1}}}$  are in  $R$ . By the definition of  $\check{B}r_n^\lambda$  in (3.7), it is obviously that the middle term in the last formula is in  $\check{B}r_n^\lambda$ . So, the last formula can be rearranged such that

$$x_{(\mathfrak{s}, u)(\mathfrak{t}, v)}^\lambda \cdot b \equiv \sum_{(\mathfrak{t}_2, v_2) \in \mathcal{I}_n(k, \lambda)} a_{(\mathfrak{t}_2, v_2)} x_{(\mathfrak{s}, u)(\mathfrak{t}_2, v_2)}^\lambda \pmod{\check{B}r_n^\lambda},$$

where  $\mathfrak{t}_2 := t_{\omega_{\pi_1}} \in Std(\lambda)$ ,  $v_2 := v_{\pi_1} \in B_{k, n}$ , and  $a_{(\mathfrak{t}_2, v_2)}$  is the corresponding coefficient of  $x_{(\mathfrak{s}, u)(\mathfrak{t}_2, v_2)}^\lambda$ . Thus, we get the precise statement (3.9).  $\square$

As a consequence of the above theorem,  $\check{B}r_n^\lambda$  is the  $R$ -module freely generated by the collection (3.7).

The new basis in (3.8) of the  $q$ -Brauer algebra can be verified to be a cellular basis in the sense of Graham and Lehrer [9] by checking conditions of the definition of cellular algebra (see Definition 1.1 in [9]) as follows:

The  $q$ -Brauer algebra  $Br_n(r^2, q^2)$  has the cell datum  $(\Lambda_n, \mathcal{I}_n, C, *)$  where

(C1)  $\Lambda_n$  is a partially ordered set with the dominance order defined in Section 2.1.

For each  $(k, \lambda) \in \Lambda_n$ ,  $\mathcal{I}_n(k, \lambda)$  is a finite set satisfying that

$$C : \coprod_{(k, \lambda) \in \Lambda_n} \mathcal{I}_n(k, \lambda) \times \mathcal{I}_n(k, \lambda) \rightarrow Br_n(N)$$

determined by the rule  $C((\mathfrak{s}, u), (\mathfrak{t}, v)) = x_{(\mathfrak{s}, u)(\mathfrak{t}, v)}^\lambda$  is injective map.

(C2) This condition follows from Theorem 3.10(1).

(C3) This condition is satisfied by Theorem 3.10(2).

(C3') This condition is obtained by applying  $*$  to the equation (3.9), we obtain

$$b^* \cdot x_{(\mathfrak{t}, v)(\mathfrak{s}, u)}^\lambda \equiv \sum_{(\mathfrak{t}_2, v_2) \in \mathcal{I}_n(k, \lambda)} a_{(\mathfrak{t}_2, v_2)} x_{(\mathfrak{t}_2, v_2)(\mathfrak{s}, u)}^\lambda \pmod{\check{B}r_n^\lambda}.$$

Now we can use the representation theory of cellular algebras for the  $q$ -Brauer algebra.

For  $(k, \lambda) \in \Lambda_n$ , the *Cell* module  $C(k, \lambda)$  in the  $q$ -Brauer algebra is called *Specht* and is defined to be the  $R$ -module freely generated by

$$(3.11) \quad \left\{ x_{(\mathfrak{t}, v)}^\lambda := m_\lambda g_{d(\mathfrak{t})} g_v + \check{B}r_n^\lambda \mid (\mathfrak{t}, v) \in \mathcal{I}_n(k, \lambda) \right\}$$

and with the right  $Br_n(r^2, q^2)$  action

$$x_{(\mathfrak{t}, v)}^\lambda b + \check{B}r_n^\lambda = \sum_{(\mathfrak{t}_1, v_1) \in \mathcal{I}_n(k, \lambda)} a_{(\mathfrak{t}_1, v_1)} x_{(\mathfrak{t}_1, v_1)}^\lambda + \check{B}r_n^\lambda \quad \text{for } b \in Br_n(r^2, q^2),$$

where the coefficients  $a_{(\mathfrak{t}_1, v_1)} \in R$ , for  $(\mathfrak{t}_1, v_1)$  in  $\mathcal{I}_n(k, \lambda)$ , are determined by the expression (3.9).

The last theorem is an analogue of that of the Hecke algebra of the symmetric group  $H_n$ . So, we refer to the set (3.8) a *Murphy basis* of the  $q$ -Brauer algebras, and to the set (3.11) a *Murphy basis* of the Specht module  $C(k, \lambda)$ . Notice that we do not know the other properties of a Murphy basis for the  $q$ -algebra so far.

**Remark 3.11.** 1. Note that the notion of Specht module  $C(k, \lambda)$  of the  $q$ -Brauer algebra, which is introduced in this paper, is compatible with the Specht module  $S^\lambda$  of the Hecke algebra of the symmetric group in (2.7). If  $q = 1$ , then it recovers the notion of Specht module of the classical Brauer algebra used in [12].

2. Let  $F$  be a field and  $\hat{r}, \hat{q}, (\hat{q} - \hat{q}^{-1})$  and  $(\hat{r} - \hat{r}^{-1})$  be units in  $F$ . The assignments  $\varphi : r \mapsto \hat{r}$  and  $\varphi : q \mapsto \hat{q}$  determine a homomorphism  $R \rightarrow F$ , giving  $F$  an  $R$ -module structure. We refer to the specialization  $Br_n(\hat{r}^2, \hat{q}^2) = Br_n(r^2, q^2) \otimes_R \mathbb{F}$  as a  $q$ -Brauer algebra over  $F$ . If  $(k, \lambda) \in \Lambda_n$  then the cell module  $C(k, \lambda) \otimes_R F$  for  $Br_n(\hat{r}^2, \hat{q}^2)$  admits a symmetric associative bilinear form which is related to the generic form (3.13) in an obvious way. Similarly, this holds true for the version  $Br_n(N)$ .

3. Whenever the context is clear and no confusion can arise, the abbreviation  $Br_n(r^2, q^2)$  will be used for  $Br_n(\hat{r}^2, \hat{q}^2)$  and similarly,  $C(k, \lambda)$  will be used for the  $Br_n(\hat{r}^2, \hat{q}^2)$ -module  $C(k, \lambda) \otimes_R \mathbb{F}$ .

4. In the case  $q = 1$  the version of Theorem 3.10 for  $Br_n(N)$  coincides with Enyang's result to the classical Brauer algebra  $D_n(N)$  (see [8], Theorem 9.1). It implies that over a field  $F$  of any characteristic the other results for the  $q$ -Brauer algebra  $Br_n(N)$  in this article recover those of the classical Brauer algebra.

The example below illustrates a basis for Specht module.

**Example 3.12.** Let  $n = 5$ ,  $k = 1$ , and  $\lambda = (2, 1)$ . If  $j, i_j$  are integers with  $1 \leq i_j \leq j \leq n - 1$ , write  $t_j = 1$  or  $t_j = s_j s_{j-1} \cdots s_{i_j}$ , so that  $B_{2,5} = \{v = t_2 t_4 \mid t_j = 1 \text{ or } t_j = s_{j,i_j}, 1 \leq i_j \leq j \text{ for } j \in \{1, 2, 4\}\}$ ;  $B_{1,5} = \{v = t_2 t_3 t_4 \mid t_j = 1 \text{ or } t_j = s_{j,i_j}, 1 \leq i_j \leq j \leq 4\}$   
 $= \{\mathbf{1}, s_2, s_{2,3}, s_{2,1}, s_{2,1}s_3, s_{2,1}s_{3,2}, s_{2,4}, s_{2,1}s_{3,4}, s_{2,1}s_{3,2}s_4, s_{2,1}s_{3,2}s_{4,3}\}$ .

Since the set of partitions  $\{\mu \mid \mu \triangleright \lambda\} = \{\mu_1 = (3), \mu_2 = (1)\}$  we obtain as follows:

With  $\mu_1 = (3)$  the Young subgroup  $S_{\mu_1} = \{\mathbf{1}, s_3, s_4, s_3 s_4, s_4 s_3, s_4 s_3 s_4\}$ ,

$Std(\mu_1) = \{\mathbf{t}^{\mu_1} = [\overline{3} \overline{4} \overline{5}]\}$  and hence

$$m_{\mu_1} = e(\mathbf{1} + g_3 + g_4 + g_3 g_4 + g_4 g_3 + g_3 g_4 g_3) = e(\mathbf{1} + g_3)(\mathbf{1} + g_4 + g_4 g_3).$$

With  $\mu_2 = (1)$  the Young subgroup  $S_{\mu_2} = \{\mathbf{1}\}$ ,  $Std(\mu_2) = \{\mathbf{t}^{\mu_2} = [\overline{5}]\}$  and  $m_{\mu_2} = e_{(2)}$ .

So by (3.9) the two-sided ideal  $\check{B}r_5^{(2,1)}$  has a basis:

$$(3.12) \quad \left\{ \begin{array}{l} x_{(s_1, u_1)(t_1, v_1)}^{\mu_1} \\ x_{(s_2, u_2)(t_2, v_2)}^{\mu_2} \end{array} \middle| \begin{array}{l} (t_1, v_1), (s_1, u_1) \in \mathcal{I}_n(l, \mu_1), \\ (t_2, v_2), (s_2, u_2) \in \mathcal{I}_n(l, \mu_2) \end{array} \right\} = \left\{ \begin{array}{l} x_{(\mathbf{1}, u_1)(\mathbf{1}, v_1)}^{\mu_1} \\ x_{(\mathbf{1}, u_2)(\mathbf{1}, v_2)}^{\mu_2} \end{array} \middle| \begin{array}{l} v_1, u_1 \in B_{1,5} \\ v_2, u_2 \in B_{2,5} \end{array} \right\}.$$

In the other hand,  $Std(\lambda) = \left\{ \mathbf{t}^\lambda = [\overline{3} \overline{4} \overline{5}], \mathbf{t}^\lambda s_4 = [\overline{3} \overline{5} \overline{4}] \right\}$  and  $m_\lambda = e(1 + g_3)$ , the basis for  $C(1, \lambda)$ , of the form displayed in (3.11), is

$$\left\{ e(1 + g_3)g_v + \check{B}r_5^{(2,1)}, e(1 + g_3)g_4 g_v + \check{B}r_5^{(2,1)} \mid v \in B_{1,5} \right\}.$$

As in Proposition 2.4 of [9], the cell module  $C(k, \lambda)$  for  $Br_n(r^2, q^2)$  admits an associative bilinear form  $\langle \cdot, \cdot \rangle_\lambda : C(k, \lambda) \times C(k, \lambda) \rightarrow R$  defined by

$$(3.13) \quad \langle x_{(t,v)}^\lambda, x_{(s,u)}^\lambda \rangle_\lambda m_\lambda \equiv x_{(t,v)}^\lambda (x_{(s,u)}^\lambda)^* \pmod{\check{B}r_n^\lambda}.$$

This means

$$\begin{aligned} \langle m_\lambda g_{d(t)} g_v + \check{B}r_n^\lambda, m_\lambda g_{d(s)} g_u + \check{B}r_n^\lambda \rangle_\lambda m_\lambda &\equiv m_\lambda g_{d(t)} g_v g_u^* g_{d(t)}^* m_\lambda + \check{B}r_n^\lambda m_\lambda \\ &\equiv m_\lambda g_{d(t)} g_v g_u^* g_{d(t)}^* m_\lambda \pmod{\check{B}r_n^\lambda}. \end{aligned}$$

**Example 3.13.** Let  $n = 3$  and  $\lambda = (1)$  so that  $\check{B}r_n^\lambda = (0)$  and  $m_\lambda = e$ . We order the basis (3.11) for the module  $C(k, \lambda)$  as  $\mathbf{v}_1 = e$ ,  $\mathbf{v}_2 = eg_2$  and  $\mathbf{v}_3 = eg_2g_1$  and, with respect to this ordered basis, the Gram matrix  $\langle \mathbf{v}_i, \mathbf{v}_j \rangle_\lambda$  of the bilinear form (3.13) is

$$\begin{bmatrix} a & rq & rq^3 \\ rq & q^2a + (q^2 - 1)rq & rq^5 \\ rq^3 & rq^5 & q^4a + (q^4 - 1)rq^3 \end{bmatrix}, \text{ where } a = \frac{r - r^{-1}}{q - q^{-1}}.$$

The determinant of the Gram matrix given above is

$$(3.14) \quad \frac{3q^5(r^2 - q^2)^2(q^4r^2 - 1)}{r^3(q^2 - 1)^3}$$

#### 4. REPRESENTATION THEORY OVER A FIELD

Using the Murphy basis of Specht modules  $C(l, \lambda)$ , we have defined the new  $F$ -bilinear form,  $\langle \cdot, \cdot \rangle_\lambda$ , for the  $q$ -Brauer algebra. This bilinear form differs from the one given in [7], Definition 4.21. In detail, instead of determining the bilinear form via the known bilinear forms, including two bilinear forms of both the Hecke algebra and the iterated inflation's algebra, our bilinear form is directly defined using the Murphy basis of Specht modules. This enable us to give explicit calculations in the proof of Theorem 4.1. Also notice that in a restriction to subalgebra  $H_n$  the new bilinear form recovers the known one for the Hecke algebra.

Using the general theory of cellular algebras we obtain some results about the representation theory of  $q$ -Brauer algebras. From now on, let  $F$  be an arbitrary field of characteristic  $p \geq 0$ . Denote

$$\text{rad}(C(k, \lambda)) = \{x \in C(k, \lambda) \mid \langle x, y \rangle_\lambda = 0 \text{ for all } y \in C(k, \lambda)\}$$

and

$$D(k, \lambda) = C(k, \lambda) / \text{rad}(C(k, \lambda)).$$

The following are special cases of results in [9].

**Statement 1.** For  $(k, \lambda) \in \Lambda_n$ ,  $r$ ,  $q$  and  $(r - r^{-1})/(q - q^{-1})$  invertible elements in an arbitrary field  $F$ , let  $Br_n(r^2, q^2)$  be a  $q$ -Brauer algebra over  $F$ . Then

- (1)  $\text{rad}(C(k, \lambda))$  is a  $Br_n(r^2, q^2)$ -submodule of  $C(k, \lambda)$ ;
- (2) If  $D(k, \lambda) \neq 0$  then
  - (a)  $D(k, \lambda)$  is simple;
  - (b)  $\text{rad}(C(k, \lambda))$  is the radical of the  $Br_n(r^2, q^2)$ -module  $C(k, \lambda)$ .

**Statement 2.** For  $(k, \lambda), (l, \mu) \in \Lambda_n$ , let  $Br_n(r^2, q^2)$  be a  $q$ -Brauer algebra over an arbitrary field  $F$ . Suppose  $M$  is a  $Br_n(r^2, q^2)$ -submodule of  $C(k, \lambda)$  and

$$\varphi : C(l, \mu) \longrightarrow C(k, \lambda) / M$$

is a  $Br_n(r^2, q^2)$ -module homomorphism, and  $\langle \cdot, \cdot \rangle_\mu \neq 0$ . Then

- (1)  $\varphi \neq 0$  only if  $\lambda \supseteq \mu$ .
- (2) If  $\lambda = \mu$ , then there are elements  $r_0 \neq 0$  and  $r_1$  in  $F$  such that for all  $x \in C(l, \mu)$ , we have  $r_0\varphi(x) = r_1x + M$ .



For  $(k, \lambda), (l, \mu) \in \Lambda_n$  and  $D(l, \mu) \neq 0$ , let  $d_{\lambda\mu} = [C(k, \lambda) : D(l, \mu)]$  be the composition multiplicity of  $D(l, \mu)$  in  $C(k, \lambda)$ .

The next statement provides a classification of the simple  $Br_n(r^2, q^2)$ -modules. This result is an analogue of that for the Hecke algebra due to Dipper and James (see [4], Theorem 7.6).

**Theorem 4.1.** *For  $(k, \lambda) \in \Lambda_n$ ,  $r, q$  and  $(r - r^{-1})/(q - q^{-1})$  invertible elements in an arbitrary field  $F$ , let  $Br_n(r^2, q^2)$  be a  $q$ -Brauer algebra over  $F$ . Then*

- (1) *The set  $\{D(l, \mu) \mid (l, \mu) \in \Lambda_n \text{ and } \mu \text{ is an } e(q^2)\text{-restricted partition}\}$  is a complete set of pairwise non-isomorphic simple  $Br_n(r^2, q^2)$ -modules.*
- (2) *For  $(k, \lambda), (l, \mu) \in \Lambda_n$ , suppose that  $\mu$  is an  $e(q^2)$ -restricted partition. Then  $d_{\mu\mu} = 1$  and  $d_{\lambda\mu} \neq 0$  only if  $\lambda \supseteq \mu$ .*

*Proof.* (1). Since the  $q$ -Brauer algebra is cellular, it follows from Theorem 3.4 [9] that the set

$$\{D(l, \mu) \mid D(l, \mu) \neq 0 \text{ for partition } \mu \text{ of } n - 2l, 0 \leq l \leq [n/2]\}$$

is a complete set of pairwise non-isomorphic simple  $Br_n(r^2, q^2)$ -modules. The remainder of proof is to show that  $D(l, \mu) \neq 0$  if and only if  $\mu$  is  $e(q^2)$ -restricted partition of  $n - 2l$ .

Indeed, pick two non-zero elements  $x_{(\mathfrak{s}, u)}^\mu = m_\mu g_{d(\mathfrak{s})} g_u + \check{B}r_n^\mu$  and  $x_{(\mathfrak{t}, v)}^\mu = m_\mu g_{\mathfrak{t}} g_v + \check{B}r_n^\mu$  in  $C(l, \mu)$  with arbitrary pairs  $(\mathfrak{s}, u), (\mathfrak{t}, v) \in \mathcal{I}_n(l, \mu)$ . This yields, using (3.13),

(4.1)

$$\begin{aligned} \langle x_{(\mathfrak{s}, u)}^\mu, x_{(\mathfrak{t}, v)}^\mu \rangle_\mu m_\mu &= \langle m_\mu g_{d(\mathfrak{s})} g_u + \check{B}r_n^\mu, m_\mu g_{d(\mathfrak{t})} g_v + \check{B}r_n^\mu \rangle_\mu m_\mu \\ &\stackrel{(3.13)}{\equiv} m_\mu g_{d(\mathfrak{s})} g_v g_u^* g_{d(\mathfrak{t})}^* m_\mu \pmod{\check{B}r_n^\mu} \\ &\stackrel{(3.5)}{\equiv} e_{(l)}(c_\mu g_{d(\mathfrak{s})}) g_v g_u^* (g_{d(\mathfrak{t})}^* c_\mu) e_{(l)} \pmod{\check{B}r_n^\mu} \\ &\stackrel{(2.5)}{\equiv} e_{(l)}(c_{1\mathfrak{s}} g_v)(g_u^* c_{1\mathfrak{t}}) e_{(l)} \pmod{\check{B}r_n^\mu} \\ &\stackrel{(2.6)}{\equiv} e_{(l)} \left( \sum_{\mathfrak{s}_1 \in Std(\mu)} a_{\mathfrak{s}_1} c_{1\mathfrak{s}_1} + \check{\mathcal{H}}_{2l+1, n}^\mu \right) \left( \sum_{\mathfrak{t}_1 \in Std(\mu)} a_{\mathfrak{t}_1} c_{1\mathfrak{t}_1} + \check{\mathcal{H}}_{2l+1, n}^\mu \right) e_{(l)} \pmod{\check{B}r_n^\mu} \\ &\stackrel{(2.7)}{\equiv} e_{(l)} \left( \sum_{\mathfrak{s}_1 \in Std(\mu)} a_{\mathfrak{s}_1} c_{\mathfrak{s}_1} \right) \left( \sum_{\mathfrak{t}_1 \in Std(\mu)} a_{\mathfrak{t}_1} c_{\mathfrak{t}_1}^* \right) e_{(l)} \pmod{\check{B}r_n^\mu} \\ &\equiv e_{(l)} \sum_{\mathfrak{s}_1, \mathfrak{t}_1 \in Std(\mu)} a_{\mathfrak{s}_1} a_{\mathfrak{t}_1} (c_{\mathfrak{s}_1} c_{\mathfrak{t}_1}^*) e_{(l)} \pmod{\check{B}r_n^\mu} \\ &\stackrel{(2.9)}{\equiv} e_{(l)} \sum_{\mathfrak{s}_1, \mathfrak{t}_1 \in Std(\mu)} a_{\mathfrak{s}_1} a_{\mathfrak{t}_1} (\langle c_{\mathfrak{s}_1}, c_{\mathfrak{t}_1} \rangle c_\mu + \check{\mathcal{H}}_{2l+1, n}^\mu) e_{(l)} \pmod{\check{B}r_n^\mu} \\ &\stackrel{(3.5), L3.3(2)}{\equiv} \sum_{\mathfrak{s}_1, \mathfrak{t}_1 \in Std(\mu)} \left( \frac{r - r^{-1}}{q - q^{-1}} \right)^l a_{\mathfrak{s}_1} a_{\mathfrak{t}_1} \langle c_{\mathfrak{s}_1}, c_{\mathfrak{t}_1} \rangle m_\mu \pmod{\check{B}r_n^\mu} \end{aligned}$$

where  $a_{\mathfrak{s}_1}, a_{\mathfrak{t}_1}$  are coefficients in  $F$ .

Then, using Theorem 2.2(1) if  $\mu$  is an  $e(q^2)$ -restricted partition of  $n - 2l$  then  $D^\mu \neq 0$ . The definition of  $D^\mu$  implies that there exist  $\mathfrak{s}, \mathfrak{t} \in Std(\mu)$  such that  $\langle c_{\mathfrak{s}}, c_{\mathfrak{t}} \rangle \neq 0$ . Now

fixing  $x_{(\mathfrak{s},1)}^\mu = m_\mu g_{d(\mathfrak{s})} + \check{B}r_n^\mu$  and  $x_{(\mathfrak{t},1)}^\mu = m_\mu g_{d(\mathfrak{t})} + \check{B}r_n^\mu$  the previous calculation yields

$$\langle x_{(\mathfrak{s},1)}^\mu, x_{(\mathfrak{t},1)}^\mu \rangle_\mu m_\mu \equiv \left( \frac{r - r^{-1}}{q - q^{-1}} \right)^l \langle c_{\mathfrak{s}}, c_{\mathfrak{t}} \rangle m_\mu \pmod{\check{B}r_n^\mu}$$

and hence  $\langle x, y \rangle_\mu = \left( \frac{r - r^{-1}}{q - q^{-1}} \right)^l \langle c_{\mathfrak{s}}, c_{\mathfrak{t}} \rangle \neq 0$ . Thus  $D(l, \mu) \neq 0$ .

Conversely, if  $\mu$  is not  $e(q^2)$ -restricted then by Theorem 2.2(1) the  $R$ -bilinear form  $\langle \cdot, \cdot \rangle$  on the Specht module  $S^\mu$  of the Hecke algebra  $H_{2k+1,n}$  is zero. This means that  $\langle c_{\mathfrak{s}_1}, c_{\mathfrak{t}_1} \rangle = 0$  for any  $c_{\mathfrak{s}_1}, c_{\mathfrak{t}_1} \in S^\mu$ . By calculation in (4.1) it implies  $\langle x_{(\mathfrak{s},u)}^\mu, x_{(\mathfrak{t},v)}^\mu \rangle_\mu = 0$  for all  $x_{(\mathfrak{s},u)}^\mu, x_{(\mathfrak{t},v)}^\mu \in C(l, \mu)$ , namely,  $D(l, \mu) = 0$ .

(2). This statement follows by applying the general theory of cellular algebras and Proposition 3.6 [9].  $\square$

**Corollary 4.2.** *For  $(k, \lambda) \in \Lambda_n$ ,  $r$ ,  $q$  and  $(r - r^{-1})/(q - q^{-1})$  invertible elements in an arbitrary field  $F$ , let  $Br_n(r^2, q^2)$  be a  $q$ -Brauer algebra over  $F$ . The following statements are equivalent.*

- (1)  $Br_n(r^2, q^2)$  is semisimple;
- (2)  $C(k, \lambda) = D(k, \lambda)$  for all  $(k, \lambda) \in \Lambda_n$ ; and,
- (3) The  $F$ -bilinear form  $\langle \cdot, \cdot \rangle_\lambda$  (cf. (3.13)) is non-degenerate for all  $(k, \lambda) \in \Lambda_n$ .

**Remark 4.3.** The same results as Theorem 4.1 and Corollary 4.2 hold true for the version  $Br_n(N)$  of the  $q$ -Brauer algebra. Furthermore, when  $q = 1$  then the statement in Theorem 4.1 recovers that for the classical Brauer algebra with non-zero parameter which was shown in [9], Theorem 4.17 by Graham and Lehrer. Also notice that in this case the cell module of the Brauer algebra in [9] is dual to the one in this paper.

## 5. IS THE $q$ -BRAUER ALGEBRA GENERICALLY ISOMORPHIC WITH THE BMW-ALGEBRA?

In this section, we answer the question whether the  $q$ -Brauer algebra is isomorphic with the BMW-algebra? Using the cellular property and explicit calculations on the Murphy basis of the Specht modules  $C(k, \lambda)$ , we show that in general the answer is "No". To this end, we need to use the following results.

**Proposition 5.1.** *Let  $Br_n(r^2, q^2)$  be the  $q$ -Brauer algebra over an arbitrary field  $F$  with invertible elements  $r$ ,  $q$  and  $\frac{r - r^{-1}}{q - q^{-1}} \in F$ . Then*

- (1)  $Br_2(r^2, q^2)$  is semisimple if and only if  $e(q^2) > 2$ .
- (2)  $Br_3(r^2, q^2)$  is semisimple if and only if  $e(q^2) > 3$  and  $\frac{3q^5(r^2 - q^2)^2(q^4r^2 - 1)}{r^3(q^2 - 1)^3} \neq 0$ .

*Proof.* If  $n = 2$  and  $\lambda$  is a partition of 2, then the cell modules  $C(0, \lambda)$  coincide with the cell modules  $S^\lambda$  of the Hecke algebra  $H_2$ , and it hence  $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_\lambda$ . Therefore, by Corollary 2.3, the  $F$ -bilinear form  $\langle \cdot, \cdot \rangle$  is non-degenerate if and only if  $e(q^2) > 2$ . If  $\lambda = \emptyset$ , then  $\check{B}r_2^\lambda \equiv \check{\mathcal{H}}_2^\lambda = F$  and  $m_\lambda = e$ . As shown in (3.11), the cell module  $C(1, \lambda)$  has a basis

$$\{ eg_v + \check{B}r_2^\lambda \mid v \in B_{1,2} = \{\mathbf{1}\} \} = \{e\}.$$

The Gram determinant with respect to this basis is  $\langle e, e \rangle_\lambda e = e^2 \stackrel{(E_1)}{=} \frac{r - r^{-1}}{q - q^{-1}} e$ , that is,  $\langle e, e \rangle_\lambda = \frac{r - r^{-1}}{q - q^{-1}}$ . With  $n = 3$  and  $\lambda$  a partition of 3, using the same argument as above yields that the  $F$ -bilinear form  $\langle \cdot, \cdot \rangle_\lambda$  is non-degenerate if and only if  $e(q^2) > 3$ . Otherwise, if  $n = 3$  and  $\lambda = (1)$  then applying Example (3.13), the Gram determinant on  $C(1, \lambda)$  is non-zero if and only if

$$\frac{3q^5(r^2 - q^2)^2(q^4r^2 - 1)}{r^3(q^2 - 1)^3} \neq 0.$$

Hence, we get the statement (2) by using Corollary 4.2.  $\square$

When replacing the version  $Br_n(r^2, q^2)$  by  $Br_n(N)$  or  $Br_n(r, q)$  used by Wenzl [23] and Dung [7], then the results are the following.

**Proposition 5.2.** *Let  $Br_n(r, q)$  be the  $q$ -Brauer algebra over an arbitrary field  $F$  with invertible elements  $r, q$  and  $\frac{r-1}{q-1} \in F$ . Then*

- (1)  $Br_2(r, q)$  is semisimple if and only if  $e(q) > 2$ .
- (2)  $Br_3(r, q)$  is semisimple if and only if  $e(q) > 3$  and  $\frac{3q(r-q)^2(q^2r-1)}{(q-1)^3} \neq 0$ .

The proof is similar to the one above, using Section 3 in [7] for calculations.

**Proposition 5.3.** *Let  $N \in \mathbb{Z} \setminus \{0\}$  and  $Br_n(N)$  be the  $q$ -Brauer algebra over an arbitrary field  $F$  with  $0 \neq q, [N] \in F$ . Then*

- (1)  $Br_2(N)$  is semisimple if and only if  $e(q) > 2$ .
- (2)  $Br_3(N)$  is semisimple if and only if  $e(q) > 3$  and

$$3q^4(q^N - q[N])([N] + q^{N+1} + q^{N+3}) \neq 0.$$

The proof uses the same arguments as in Proposition 5.1, using definition of  $Br_n(N)$  given in Remark 3.2(2) for calculations.

**Remark 5.4.** 1. Notice that if  $q = 1$  then  $e(q^2)$  (resp.  $e(q)$ ) is equal to the characteristic  $p$  of the field  $F$ . It implies that for  $r = q^N$  with  $N \in \mathbb{Z} \setminus \{0\}$  and the limit  $q \rightarrow 1$ , our results above recover these ones for the classical Brauer algebra  $D_n(N)$  due to Rui [20] in the case  $n \in \{2, 3\}$ . In particular, when  $\lim_{q \rightarrow 1} \frac{r - r^{-1}}{q - q^{-1}} = N$  and

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{3q^5(r^2 - q^2)^2(q^4r^2 - 1)}{r^3(q^2 - 1)^3} &= \lim_{q \rightarrow 1} \frac{3q^5(q^{2N} - q^2)^2(q^4q^{2N} - 1)}{q^{3N}(q^2 - 1)^3} \\ &= \lim_{q \rightarrow 1} \frac{3q^9}{q^{3N}} \cdot \frac{(q^{2(N-1)} - 1)^2}{(q^2 - 1)^2} \cdot \frac{(q^{2(N+2)} - 1)}{(q^2 - 1)} = 3(N-1)(N+2), \end{aligned}$$

then  $Br_n(q^{2N}, q^2) \equiv D_n(N)$  over the field  $F$  in which the limit  $q \rightarrow 1$  can be formed. Applying Proposition 5.1 it implies the following: Over the complex field  $Br_2(q^{2N}, q^2)$  is semisimple if and only if  $N \neq 0$  and  $Br_3(q^{2N}, q^2)$  is semisimple if and only if  $N \notin \{-2, 0, 1\}$ ; over arbitrary field of characteristic  $p > 0$ ,  $Br_2(q^{2N}, q^2)$  is semisimple if and only if  $N \neq 0$  and  $p > 2$ , and  $Br_3(q^{2N}, q^2)$  is semisimple if and only if  $N \notin \{-2, 0, 1\}$  and  $p > 3$ . These

imply that in the limit  $q \rightarrow 1$  Proposition 5.1 recovers Theorems 1.2(a) and 1.3(a) in [20] for  $n \in \{2, 3\}$ . The other computation for the version  $Br_n(r, q)$  is left to the reader.

Similarly, in the case  $q = 1$ ,  $Br_n(N)$  coincides with the classical Brauer algebra  $D_n(N)$  over arbitrary field  $F$ ,  $\text{char} F = p \geq 0$ , and direct calculation yields that Proposition 5.3 recovers Theorems 1.2(a) and 1.3(a) for  $n \in \{2, 3\}$  in [20].

2. Over the field of characteristic zero all results above agree with Wenzl's results for  $n \in \{2, 3\}$  (see Theorem 5.3 [22]). In particular, for  $Br_n(r^2, q^2)$  (resp.  $Br_n(r, q)$ ) the pair of parameters  $(\xi, \rho)$  in his theorem is replaced by  $(q^2, r)$  (resp.  $(q, r)$ ), respectively. And for  $Br_n(N)$ ,  $(\xi, \rho)$  is replaced by  $(q^2, q^N)$ .

3. Propositions 5.1 and 5.2 imply a negative answer for the question about the existence of an isomorphism between the  $q$ -Brauer algebra  $Br_n(r^2, q^2)$  (resp.  $Br_n(r, q)$ ) and the BMW-algebra  $\mathcal{B}_n$ . Three following examples illustrate Claim 1.1.

*In the two following examples, with a same parameter value the BMW-algebra is not simple, but the  $q$ -Brauer algebra is semisimple*

**Example 5.5.** We consider both algebras  $Br_3(r^2, q^2)$  and  $\mathcal{B}_3$  over the complex field. These algebras simultaneously depend on two parameters  $r$  and  $q$ . Fixing  $r = q^{-1}$  and  $q^2 = -i$ , then by Theorem 5.9(b) [19] the BMW-algebra  $\mathcal{B}_3$  is not semisimple since

$$q^4 + 1 = (-i)^2 + 1 = 0.$$

On the other hand, both  $[m]_{q^2} = 1 + q^2 = 1 - i \neq 0$  and

$$[m]_{q^2} = 1 + q^2 + (q^2)^2 = 1 - i + (-i)^2 = -i \neq 0, \text{ namely, } e(q^2) = m > 3.$$

Moreover, a direct calculation yields

$$\frac{3q^5(r^2 - q^2)^2(q^4r^2 - 1)}{r^3(q^2 - 1)^3} = \frac{3q^5(q^{-2} - q^2)^2(q^4q^{-2} - 1)}{q^{-3}(q^2 - 1)^3} = 6i \neq 0.$$

Therefore, applying Proposition 5.1(2) the  $q$ -Brauer algebra  $Br_3(r^2, q^2)$  is semisimple.

**Example 5.6.** With respect to the version  $Br_3(r, q)$  and  $\mathcal{B}_3$  over the complex field, we choose  $r = q^{-1}$  and  $q = i\sqrt{i}$ , then by Theorem 5.9(b) [19] the BMW-algebra  $\mathcal{B}_3$  is not semisimple since  $q^4 + 1 = (i\sqrt{i})^4 + 1 = 0$ .

In other words, both  $[m]_q = 1 + q = 1 + i\sqrt{i} \neq 0$  and

$$[m]_q = 1 + q + q^2 = 1 + i\sqrt{i} + (i\sqrt{i})^2 = i\sqrt{i} \neq 0, \text{ namely, } e(q) = m > 3.$$

By direct calculation, it yields

$$\frac{3q(r - q)^2(q^2r - 1)}{(q - 1)^3} = \frac{3q(q^{-1} - q)^2(q^2q^{-1} - 1)}{(q - 1)^3} = 3q^{-1} = 3(i\sqrt{i})^{-1} \neq 0.$$

Hence, by Proposition 5.2(2) the  $q$ -Brauer algebra  $Br_3(r, q)$  is semisimple.

The result is illustrated in the following table:

$\mathbb{C}$	BMW-algebra	$q$ -Brauer algebra
$(r, q^2) = (q^{-1}, -i)$	$\mathcal{B}_3$ is not semisimple	$Br_3(r^2, q^2)$ is semisimple
$(r, q) = (q^{-1}, i\sqrt{i})$	$\mathcal{B}_3$ is not semisimple	$Br_3(r, q)$ is semisimple

The next example shows that over the field of characteristic  $p = 5$  the BMW-algebra is not semisimple with total twelve parameter values, but the  $q$ -Brauer algebra is not semisimple with less than four parameter values.

**Example 5.7.** Over the prime field  $\mathbb{F}_5$  if  $q \in \{\bar{2}, \bar{3}\}$ , then it is obvious that  $[m]_{q^2} = \bar{1} + q^2 = \bar{0}$  and hence  $e(q^2) \leq 2$ . Applying Theorem 5.9 in [20] the BMW-algebra  $\mathcal{B}_2$  is not semisimple for all  $r \in \mathbb{F}_5 \setminus \{\bar{0}\}$ . Otherwise, with  $q \in \mathbb{F}_5 \setminus \{\bar{0}, \bar{2}, \bar{3}\}$  a direct calculation implies that  $e(q^2) > 2$ , and by Theorem 5.9 [20]  $\mathcal{B}_2$  is not semisimple for  $r \in \{q^{-1}, -q\} = \{\bar{1}, \bar{4}\}$ . Thus, there totally exist twelve value pairs  $(r, q)$  such that the BMW-algebra  $\mathcal{B}_2$  is not semisimple over the field  $\mathbb{F}_5$ .

By Proposition 5.1(1) the  $q$ -Brauer algebra  $Br_2(r^2, q^2)$  over the field  $\mathbb{F}_5$  is not semisimple if and only if  $q \in \{\bar{2}, \bar{3}\}$  and  $r \in \mathbb{F}_5 \setminus \{\bar{0}\}$  such that  $(r - r^{-1})/(q - q^{-1}) \neq 0$ . Direct calculation yields  $Br_2(r^2, q^2)$  over the field  $\mathbb{F}_5$  is not semisimple for all parameters  $q \in \{\bar{2}, \bar{3}\}$  and  $r \in \{\bar{2}, \bar{3}\}$ . This means that there are totally such four value pairs  $(r, q)$ .

Similarly, on the version  $Br_n(r, q)$  Proposition 5.2(1) implies that the  $q$ -Brauer algebra  $Br_2(r, q)$  over the field  $\mathbb{F}_5$  is not semisimple if and only if  $q \in \{\bar{4}\}$  and  $r \in \mathbb{F}_5 \setminus \{\bar{0}\}$  such that  $(r - 1)/(q - 1) \neq 0$ . That is,  $Br_2(r, q)$  over the field  $\mathbb{F}_5$  is not semisimple for all parameters  $q = \bar{4}$  and  $r \in \{\bar{2}, \bar{3}, \bar{4}\}$ .

The total parameter values, such that the algebras are not semisimple, are summarized in the following table.

The non-semisimple case	$\mathbb{F}_5 \times \mathbb{F}_5$
The BMW-algebra $\mathcal{B}_2$	$(r, q) \in (\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\} \times \{\bar{2}, \bar{3}\}) \cup (\{\bar{2}, \bar{3}\} \times \{\bar{1}, \bar{4}\})$
The $q$ -Brauer algebra $Br_2(r^2, q^2)$	$(r, q) \in \{\bar{2}, \bar{3}\} \times \{\bar{2}, \bar{3}\}$
The $q$ -Brauer algebra $Br_2(r, q)$	$(r, q) \in \{\bar{2}, \bar{3}, \bar{4}\} \times \{\bar{4}\}$

Thus, these examples imply that in general there does not exist an algebra isomorphism between the  $q$ -Brauer algebra and the BMW-algebra.

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